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Orthogonal polynomials and a Dirichlet problem related to the Hilbert transform

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ABSTRACT

An operator closely related to the Hilbert transform on the circle is shown to be unitarily equivalent to a shift realized on a basis of Pollaczek polynomials, a family of orthogonal polynomials with weight supported by all of the real line. There is an associated Dirichlet problem for the disk, where one wants to find harmonic functions with specified boundary values on the upper half circle and with specified constant (possibly complex) direction of the derivative on the real diameter. The Poisson kernel is found and is used to obtain continuous and L^p function existence and boundedness results. The Dirichlet problem is a limiting case of boundary value problems for certain special functions coming from the Heisenberg groups.

INTRODUCTION

An operator closely related to the Hilbert transform on the circle is shown to be unitarily equivalent to a shift realized on a basis of Pollaczek polynomials. This operator is the composition of the Hilbert transform H restricted to the even L^2-functions followed by the multiplication M by sgn θ, on the circle $-\pi<\theta\leq\pi$. The polynomials are in one of the families investigated by Pollaczek [8], with an orthogonality structure on the whole real line. The study of the operator MH leads directly to a Dirichlet problem for the disk on which one wishes to find harmonic functions with specified boundary values on the upper half circle $(0<\theta<\pi)$ and with the derivative on the real diameter being of

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specified constant (possibly complex) direction, see below. The prototype for the problem is the class of even ($f(re^{i\theta})=f(re^{-i\theta})$) harmonic functions. Our modification is a limiting case of a Dirichlet problem for certain special functions coming from the Heisenberg group.

Fundamental to the analysis is a group of Möbius transformations which commutes with the operator MH and leaves the directional derivative condition invariant.

After a summary of results, the development of the paper is divided into three sections and an appendix.

§ 1: the analysis of MH, its structure as a left shift, a one-parameter family of rational functions of MH, each of which acts as a shift on a certain weighted L^2-space having a family of Pollaczek polynomials as basis;

§ 2: the Dirichlet problem for $L^2(0,\pi)$, sketch of the related problem on the Heisenberg group, use of the Pollaczek polynomials to find an explicit solution to the following problem: fix a complex number μ, for each $f\in L^2(0,\pi)$ find coefficients $\{a_n : n\geq 0\}$ such that the harmonic function

$$\sum_{n=0}^{\infty} a_n r^n(\cos n\theta + i\mu \sin n\theta) \text{ (for which } \frac{\partial}{\partial y}=i\mu\frac{\partial}{\partial x} \text{ on } \mathbb{R})$$

has f as boundary value on $0<\theta<\pi$ as $r\to 1-$; of course the possibility of such a solution depends on μ;

§ 3: the Poisson kernel is explicitly found and is used to prove existence and convergence theorems for Poisson integrals of continuous or L^p functions on $(0,\pi)$. The permissible values of μ depend on p. For continuous functions only the values $\mu\geq 1$ and $\mu\leq -1$ are excluded;

Appendix: a summary of needed results about the Pollaczek polynomials with orthogonality on the whole real line.

The following measures and L^p spaces will occur in the paper:

$L^p(0,\pi)$: the space $L^p((0,\pi);(1/\pi)d\theta)$, at times it is convenient to identify $(0,\pi)$ with the upper half circle $\{e^{i\theta}:0<\theta<\pi\}$;

$L^p(T)$: on the circle $T:=\{e^{i\theta}:-\pi<\theta\leq\pi\}$, with measure $(1/2\pi)d\theta$;

L_E: the subspace of $L^2(T)$ consisting of even ($f(e^{-i\theta})=f(e^{i\theta})$) functions, sometimes identified in the obvious way with $L^2(0,\pi)$;

μ_β: for $-\pi/2<\beta<\pi/2$ the measure

$$d\mu_\beta(s):=\frac{\cos\beta}{2}\frac{e^{\beta s}}{\operatorname{ch}\frac{1}{2}\pi s}\,ds, \text{ on } \mathbb{R};$$

$L^p(\mu_\beta)$: short for $L^p(\mathbb{R},\mu_\beta)$.

Crucial also is the function

$$w(\theta):=-(2/\pi)\log|\tan\theta/2|$$

(whose harmonic conjugate is $\operatorname{sgn}\theta$), which maps $(0,\pi)$ onto $\mathbb{R}$ and induces an isomorphism of $L^p(\mu_0)$ to $L^p(0,\pi)$. The orthonormal polynomials for μ_0 will be denoted by $p_n(x)$; they are sometimes called Meixner-Pollaczek polynomials, see the Appendix for more details. They are generated by

$$(1+it)^{-\frac{1}{2}-\frac{1}{2}ix}(1-it)^{-\frac{1}{2}+\frac{1}{2}ix}=\sum_{n=0}^{\infty} t^n p_n(x) \quad (|t|<1).$$

Define two self-adjoint operators on $L^2(T)$ by

$$He^{in\theta}:=(\text{sgn } n)e^{in\theta}(n\in Z),$$

the Hilbert transform, and

$$Mf(\theta):=(\text{sgn } \theta)f(\theta), (-\pi<\theta<\pi),$$

the "multiplier".

THEOREM 1. *HM is an isometry on L_E and is a (right) shift of multiplicity* 1 *with cokernel* $\mathbb{C}1$ (*the constant functions*). *Specifically*

$$HMp_n(w)=-ip_{n+1}(w), \quad n\geq 0,$$

$$MHp_n(w)=ip_{n-1}(w), \qquad n\geq 1,$$

and $\{p_n(w): n\geq 0\}$ *is an orthonormal basis for* L_E.

For a complex number λ say that a C^∞-function on the disk $D:=\{\zeta\in\mathbb{C}: |\zeta|<1\}$ or on the strip $S:=\{\zeta\in\mathbb{C}: |\text{Im } \zeta|<1\}$ *satisfies condition* CR_λ (suggesting "Cauchy-Riemann") if f is harmonic and

$$e^{-\lambda}\frac{\partial f}{\partial\zeta}(x)=e^{\lambda}\frac{\partial f}{\partial\bar{\zeta}}(x)$$

for x real in the disk or strip, respectively (equivalently,

$$(\text{ch } \lambda)\frac{\partial f}{\partial y}(x)-(i \text{ sh } \lambda)\frac{\partial f}{\partial x}(x)=0).$$

In particular, condition CR_0 is equivalent to being even and harmonic, and in a limiting sense $CR_{+\infty}$ is analyticity. We consider the problem of finding functions satisfying CR_λ with specified $L^p(0,\pi)$ or $C[0,\pi]$ boundary values in the upper half disk. We will show by finding the Poisson kernel that this problem has unique and appropriately bounded solutions for any L^p, $1\leq p<\infty$ when $|\text{Im } \lambda|<\frac{1}{2}\pi\ (1-1/p)$ for $p>1$, $\text{Im } \lambda=0$ for $p=1$. The device to study CR_λ is the operator MH, since $I+(\text{th } \lambda)MH$ maps L_E (condition CR_0) into L^2 functions satisfying CR_λ (in a sense to be made precise later).

The subgroup G of the full Möbius group of the disk that preserves the upper half disk leaves the property CR_λ invariant, and commutes with MH on L_E. Indeed this group consists of the transformations

$$f_t(\zeta):=\frac{\zeta+\text{th } \frac{1}{4}\pi t}{(\text{th } \frac{1}{4}\pi t)\zeta+1}, \quad t\in\mathbb{R},$$

and is isomorphic to $(\mathbb{R},+)$ since $f_{t_1}\circ f_{t_2}=f_{t_1+t_2}$. Accordingly all integral transforms of interest in this paper will be expressed as convolution over G. The *conformal map* (to be used throughout)

$$\varrho(\zeta):=\frac{2}{\pi}\log\frac{1+\zeta}{1-\zeta},$$

which maps the disk onto the strip $S=\{\zeta:|\mathrm{Im}\ \zeta|<1\}$, takes the action of f_t into translation by t, that is, $\varrho(f_t(\zeta))=\varrho(\zeta)+t$ ($|\zeta|<1$, $t\in\mathbb{R}$). We will denote the inverse mapping of ϱ by

$$W(\sigma+i\tau):=\mathrm{th}\ \tfrac{1}{4}\pi(\sigma+i\tau),\quad (\sigma\in\mathbb{R},\ -1\leq\tau\leq 1).$$

Note that $W(\sigma+i)=e^{i\theta}$ with $\cos\theta=\mathrm{th}\ \frac{1}{2}\pi\sigma$, and $\lim_{r\to 1-}\mathrm{Re}\ \varrho(re^{i\theta})=w(\theta)$ for $0<\theta<\pi$. The lines $\tau=$constant are mapped by W to circular arcs joining -1 to $+1$ in the disk.

DEFINITION. For $\lambda\in\mathbb{C}$, $\sigma+i\tau\in S$ let

$$K_\lambda(\sigma,\tau):=\frac{e^{i\lambda\sigma}}{4|\mathrm{ch}\ \frac{1}{2}\pi(\sigma+i\tau)|^2}\{e^{\pi\sigma/2}\sin((\tfrac{1}{2}\pi-i\lambda)(1-\tau))$$
$$+e^{-\pi\sigma/2}\sin((\tfrac{1}{2}\pi+i\lambda)(1-\tau))\}.$$

For a measurable function f on $(0,\pi)$ satisfying

$$\int_0^\pi|f(\theta)|(\tan\theta/2)^{-2(\mathrm{Im}\ \lambda)/\pi}d\theta<\infty$$

the Poisson integral is defined by

$$P_\lambda[f](W(\sigma+i\tau)):=\int_{\mathbb{R}}f(\arccos\ \mathrm{th}\ \tfrac{1}{2}\pi w)K_\lambda(\sigma-w,\tau)dw.$$

THEOREM 2. *For $\lambda\in\mathbb{R}$, $1\leq p\leq\infty$, $f\in L^p(0,\pi)$ the Poisson integral $P_\lambda[f]$ satisfies CR_λ and converges radially to f, that is,*

$$\int_0^\pi|P_\lambda[f](re^{i\theta})-f(\theta)|^pd\theta\to 0 \text{ as } r\to 1-,$$

for $1\leq p<\infty$. Further if $f\in C[0,\pi]$ then $P_\lambda[f]$ extends to a continuous function on the closed upper half disk and agrees with f on the upper half circle.

For $\mathrm{Im}\ \lambda\neq 0$ there is a modified L^p boundedness and convergence.

THEOREM 3. *For $\lambda\in\mathbb{C}$, $1<p<\infty$, $|\mathrm{Im}\ \lambda|<\frac{1}{2}\pi(1-1/p)$, if $f\in L^p(0,\pi)$ then*

$$\int_{\mathbb{R}}|P_\lambda[f](W(\sigma+i\tau))F_{\tau,\lambda}(\sigma)^{1/p}-f(W(\sigma+i))|^pd\mu_0(\sigma)\to 0 \text{ as } \tau\to 1-,$$

where

$$F_{\tau,\lambda}(\sigma):=\exp\left\{(\mathrm{Im}\ \lambda)\left(\frac{1}{\tau}-1\right)\sigma\right\}\frac{\mathrm{ch}\ \frac{1}{2}\pi\sigma}{\tau\ \mathrm{ch}\ \dfrac{\pi\sigma}{2\tau}}\quad(0<\tau\leq 1).$$

§ 1. THE OPERATOR MH

The description of the action of MH, and its adjoint HM, is based on elementary algebraic lemmas depending on the fact that $(\varrho-\bar{\varrho})^2$ is essentially constant as a boundary value function.

In this section we use L_0 to denote the subspace of odd $(f(e^{-i\theta})=-f(e^{i\theta}))$ functions of $L^2(T)$, and $H^2(T)$ to denote the usual Hardy space ($\cong$ closed span $\{e^{in\theta}: n\geq\theta\}\subset L^2(T)$).

1.1. DEFINITION. For f analytic on D or S let $Jf(z) := \overline{f(\bar{z})}$, so that $\overline{Jf}$ is analytic, and if $f\in H^2(T)$ then $\overline{Jf}\in H^2(T)$.

1.2. PROPOSITION. *For each $f\in L^2(T)$ there are functions $g_1, g_2\in H^2(T)$ such that $f=g_1+Jg_2$; further if $f\in L_E$ then $f=g+Jg$ for some $g\in H^2$.*

1.3. PROPOSITION. *There is a nonunitary continuous representation R of G on $L^2(T)$ satisfying $\|R_tf\|_2\leq e^{\frac{1}{2}\pi|t|}\|f\|_2$ and having $H^2(T)$, L_E, and L_0 as invariant subspaces.*

PROOF. Define $R_tf(z) := f(f_t(z))$ for $|z|\leq 1$; in $|z|<1$ this is well defined for f or $\bar{f}$ in H^2, on $|z|=1$ it is defined for almost all z by using boundary values. Calculus shows that the Jacobian for the transformation f_t is bounded by $(1+|\text{th}\ \frac{1}{4}\pi t|)/(1-|\text{th}\ \frac{1}{4}\pi t|)=\exp(\frac{1}{2}\pi|t|)$. Clearly H^2 is invariant. Also $f_t(\bar{z})= =\overline{f_t(z)}$ implying that J commutes with R_t (each $t\in\mathbb{R}$), and so L_E and L_0 are invariant subspaces. □

We describe the algebraic and G-commutation properties of M and H.

1.4. PROPOSITION. *M is self-adjoint, $M^2=I$, $ML_E=L_0$ and $ML_0=L_E$. Also $MR_t=R_tM$ for all $t\in\mathbb{R}$.*

PROOF. In general, R_t commutes with multiplication $(R_t(fg)=(R_tf)(R_tg))$ and $R_t(\text{sgn}\ \theta)=\text{sgn}\ \theta$ since R_t preserves the intervals $0<\theta<\pi$ and $-\pi<\theta<0$. □

1.5. PROPOSITION. *H is self-adjoint; $H^2=P_0$ (the projection of L^2 on $1^{\perp}$), $HL_E=L_0$ and $HL_0\subset L_E$, and for $f\in L^2(T)$, $HR_tf=R_tHf$ for all $t\in\mathbb{R}$ if and only if $f\in L_E$.*

PROOF. Only the commutation relation needs any detail here. Write $f\in L^2(T)$ as $f=g+Jh$ with $g, h\in H^2(T)$. Observe $Hg=P_0g=g-g(0)$ and $H(Jh)= =-P_0(Jh)$. Then $(HR_t-R_tH)f=(g(0)-h(0)-g(\tau)+h(\tau))1$, where $\tau=\text{th}\ \frac{1}{4}\pi t$, since $Jh(x)=h(x)$ for $-1<x<1$. This is zero for all $t\in\mathbb{R}$ if and only if the analytic function $g-h$ is constant on the real diameter $-1<x<1$, if and only if $h=g+c$ for some constant c. □

1.6. PROPOSITION. *$HM|L_E$ is an isometry of codimension* 1 *and* $MH^2M|L_E=I$. *Also R_t commutes with $MH|L_E$.*

PROOF. Let $f\in L_E$ then $\langle MH^2Mf,f\rangle=\langle H^2Mf,Mf\rangle=\langle Mf,Mf\rangle=\langle f,f\rangle$ since $Mf\in L_0$ and $H^2|L_0=I$. The adjoint of HM, namely MH, has kernel $\mathbb{C}1$ since $\|MHf\|=\|Hf\|$ for each $f\in L^2(T)$. Note if $f\in L_E$ then $R_tHf=HR_tf$ (by Proposition 1.5) so that $MHR_tf=MR_t(Hf)=R_tM(Hf)$. □

General operator theory applied to HM implies that L_E splits into two closed subspaces on which HM acts respectively as a shift and as a unitary operator. We will show that there is no unitary part. The shift structure is obtained by the standard method.

1.7. PROPOSITION. *$\{(HM)^n1:n=0,1,2,\ldots\}$ is an orthonormal set in L_E and $MH(HM)^n1=(HM)^{n-1}1$ for $n\geq 1$.*

PROOF. Since HM is an isometry $\|(HM)^n1\|=1$. Let m,n be integers with $m>n\geq 0$, then $\langle (HM)^m1,(HM)^n1\rangle=\langle (HM)^n(HM)^{m-n}1,(HM)^n1\rangle=$ $=\langle (HM)^{m-n}1,(MH)^n(HM)^n1\rangle=\langle (HM)^{m-n}1,1\rangle=\langle M(HM)^{m-n-1}1,H1\rangle=0$, since $MH^2M=I$ on L_E. Similarly $MH(HM)^n1=MH^2M(HM)^{n-1}1=(HM)^{n-1}1$ for $n\geq 1$. □

We now show that w induces an isomorphism of L_E onto $L^2(\mu_0)$ which maps $\{(HM)^n1:n\geq 0\}$ to a basis of orthogonal polynomials.

1.8. LEMMA. *Any polynomial in w, respectively ϱ, is an element of L_E, respectively $H^2(T)$. Further the boundary value of* Re (ϱ^n) *is*

$$\sum_{j=0}^{[n/2]}\binom{n}{2j}(-1)^j w^{n-2j}.$$

PROOF. For any polynomial p we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|p(w(\theta))|^2d\theta=\tfrac{1}{2}\int_{\mathbb{R}}|p(w)|^2(\mathrm{ch}\ \tfrac{1}{2}\pi w)^{-1}dw<\infty.$$

Let $n\geq 1$, then

$$\mathrm{Re}\ (\varrho^n)=\mathrm{Re}\ ((\mathrm{Re}\ \varrho+i\ \mathrm{Im}\ \varrho)^n)=\sum_{j=0}^{[n/2]}\binom{n}{2j}(\mathrm{Re}\ \varrho)^{n-2j}(i\ \mathrm{Im}\ \varrho)^{2j}(-1)^j$$

in the open disk. Now take radial limits (except at ± 1), then Re $\varrho\to w$ and $(\mathrm{Im}\ \varrho)^2\to 1$. Thus Re $(\varrho^n)\in L^2(T)$ implying that $\varrho^n\in H^2(T)$. □

We will use the following to show that HM maps any polynomial in w to another polynomial in w of degree higher by one.

1.9. LEMMA. *There are constants c_{nj} defined for integers n, j with $n \geq 0$ and $1 \leq j \leq [n/2]$ ($c_{nj} = 0$ for j out of range) such that*

$$(x-y)(x^n+y^n) = \frac{2}{n+1}(x^{n+1}-y^{n+1}) + \sum_{j=1}^{[n/2]} c_{nj}(x^{n+1-2j}-y^{n+1-2j})(x-y)^{2j},$$

for any (commuting) variables x, y.

PROOF. Fix $n \geq 0$. The sets

$$E_1 := \{(x^{n+1-2j}-y^{n+1-2j})(x-y)^{2j} : 0 \leq j \leq [n/2]\}$$

and

$$E_2 := \{(x+y)^{n-2j}(x-y)^{2j+1} : 0 \leq j \leq [n/2]\}$$

are both bases for the space of skew-symmetric polynomials of degree $n+1$ in x and y. The transformation matrix from E_1 to E_2 is triangular, indeed

$$(x^{n+1-2j}-y^{n+1-2j})(x-y)^{2j} = 2^{2j-n} \sum_{k=j}^{[n/2]} \binom{n+1-2j}{2k+1-2j}(x+y)^{n-2k}(x-y)^{2k+1},$$

(this comes from expanding x^m and y^m as $2^{-m}[(x+y) \mp (x-y)]^m$ respectively). On the other hand

$$(x-y)(x^n+y^n) = 2^{1-n} \sum_{j=0}^{[n/2]} \binom{n}{2j}(x+y)^{n-2j}(x-y)^{2j+1}.$$

This has the coefficient 2^{1-n} for the $j=0$ element of E_2, which in turn appears only in the $j=0$ element of E_1, with coefficient $2^{-n}(n+1)$. Hence the $j=0$ element of E_1 appears with a coefficient of $2^{1-n}/(2^{-n}(n+1)) = 2/(n+1)$ in the expansion of $(x-y)(x^n+y^n)$. $\square$

1.10. PROPOSITION. *For $n = 0, 1, 2, \ldots$*

$$HM(\mathrm{Re}\,(\varrho^n)) = -i(n+1)\mathrm{Re}\,(\varrho^{n+1}) - \tfrac{1}{2}i \sum_{j=1}^{[n/2]} (-4)^j c_{nj} \mathrm{Re}\,(\varrho^{n+1-2j})$$

(the latter sum is vacuous if $n = 0$ or 1).

PROOF. $M(\mathrm{Re}\,(\varrho^n))$ is the boundary value of $(1/4i)(\varrho - \bar{\varrho})(\varrho^n + \bar{\varrho}^n)$ which equals

$$\frac{1}{4i}\left(\frac{2}{n+1}(\varrho^{n+1}-\bar{\varrho}^{n+1}) + \sum_{j=1}^{[n/2]} c_{nj}(\varrho^{n+1-2j}-\bar{\varrho}^{n+1-2j})(\varrho-\bar{\varrho})^{2j}\right).$$

This has the same boundary value as

$$\frac{1}{4i}\left(\frac{2}{n+1}(\varrho^{n+1}-\bar{\varrho}^{n+1}) + \sum_{j=1}^{[n/2]} c_{nj}(-4)^j(\varrho^{n+1-2j}-\bar{\varrho}^{n+1-2j})\right).$$

Now we apply H to this sum and obtain

$$\frac{1}{4i}\left(\frac{2}{n+1}(\varrho^{n+1}+\bar{\varrho}^{n+1})+\sum_{j=1}^{[n/2]} c_{nj}(-4)^j(\varrho^{n+1-2j}+\bar{\varrho}^{n+1-2j})\right),$$

the stated expression. We used the facts $H\varrho^m=\varrho^m$ and $H\bar{\varrho}^m=-\bar{\varrho}^m$ for $m\geq 1$ (since $\varrho(0)=0$), and all powers of ϱ, $\bar{\varrho}$ occurring in the expression are positive. □

1.11. THEOREM. *HM is a (right) shift of multiplicity 1 on L_E, and $\{p_n(w):n\geq 0\}$ is an orthonormal basis for L_E on which HM acts by $HMp_n(w)=-ip_{n+1}(w)$ and $MHp_n(w)=ip_{n-1}(w)$.*

PROOF. Let q_1 be a real polynomial (that is, real coefficients) of degree n with leading coefficient $c\neq 0$, then by Lemma 1.8 there is a real polynomial q_2 with the same degree and leading coefficient as q_1 such that $q_1(w)=\text{Re}\,(q_2(\varrho))$. By Proposition 1.10,

$$HMq_1(w)=HM(\text{Re}\,(c\varrho^n+q_3(\varrho)))=-\frac{ic}{n+1}\,\text{Re}\,(\varrho^{n+1})+i\text{Re}\,(q_4(\varrho))$$

$$=-\frac{ic}{n+1}\,w^{n+1}+iq_5(w),$$

where q_3, q_4, q_5 are real polynomials of degree $\leq n-1$, n, n respectively. Thus $(HM)^n 1$ is a polynomial of degree n in w with leading coefficient $(-i)^n/n!$ which is orthogonal in $L^2(\mu_0)$ to all polynomials of lower degree, by Proposition 1.7. By the uniqueness of orthogonal polynomials it must be a scalar multiple of $p_n(w)$. The latter has leading coefficient $1/n!$ (see the Appendix), hence $(HM)^n 1=(-i)^n p_n(w)$. The fact that $\{p_n(w):n\geq 0\}$ is a basis for $L^2(\mu_0)$ follows from a general theorem of Hamburger (see Freud [3], p. 84) asserting that the set of polynomials is dense in $L^2(\mathbb{R},\mu)$ for any positive measure μ such that $\int_{\mathbb{R}} e^{c|x|}d\mu(x)<\infty$ for some $c>0$. □

The theorem allows another proof of the commutativity $R_tMH=MHR_t$ on L_E. Indeed

$$R_tp_n(w)=p_n(w+t)=\sum_{j=0}^{n} p_{n-j}(w)q_j(t)$$

(see the Appendix). Now apply MH to both sides and obtain

$$i\sum_{j=0}^{n-1} p_{n-j-1}(w)q_j(t)=ip_{n-1}(w+t)=R_tMHp_n(w)$$

for the right side ($t\in\mathbb{R}$). This is a manifestation of umbral calculus and the fact that $\{n!p_n(w)\}$ is a Sheffer set (see Rota [9] Ch. 2).

There is a family of measures relevant to the Poisson integral defined in the introduction, namely $(\tan\,\theta/2)^{-2\beta/\pi}d\theta$ on $0<\theta<\pi$, or equivalently $e^{\beta w}(\text{ch}\,\frac{1}{2}\pi w)^{-1}\,dw$ on $\mathbb{R}$ (that is, μ_β) with $-\frac{1}{2}\pi<\beta<\frac{1}{2}\pi$. Pollaczek [8] found the orthogonal polynomials for these measures also. They too have a shift structure, this time for a rational function of MH. It is pointed out in Rota ([9], p. 43) that these Pollaczek polynomials have such a structure.

Any formal power series in MH may be applied to polynomials in w since MH lowers the degree. In particular we may consider

$$T_\beta := MH(I+i(\tan\beta)MH)^{-1} = \sum_{n=0}^{\infty} (-i\tan\beta)^n (MH)^{n+1}.$$

The polynomials $p_n^\beta(w)$ are described in the Appendix.

1.12. PROPOSITION. *$T_\beta p_n^\beta(w) = ip_{n-1}^\beta(w)$, for $n \geq 1$.*

PROOF. For $\alpha \in \mathbb{R}$,

$$MHe^{i\alpha w} = (1/\operatorname{ch}\alpha) \sum_{n=0}^{\infty} (i \operatorname{th}\alpha)^n MHp_n(w)$$

$$= (i/\operatorname{ch}\alpha) \sum_{n=1}^{\infty} (i\operatorname{th}\alpha)^n p_{n-1}(w) = -(\operatorname{th}\alpha)e^{i\alpha w}.$$

Thus the series

$$T_\beta e^{i\alpha w} = \sum_{n=0}^{\infty} (-i\tan\beta)^n(-\operatorname{th}\alpha)^{n+1}e^{i\alpha w} = \frac{-\operatorname{th}\alpha}{1-i\operatorname{th}\alpha\tan\beta}e^{i\alpha w}$$

converges for α in some neighborhood of 0. On the other hand

$$e^{i\alpha w}\frac{\operatorname{ch}(\alpha-i\beta)}{\cos\beta} = \sum_{n=0}^{\infty}\left(\frac{i\operatorname{th}\alpha}{1-i\operatorname{th}\alpha\tan\beta}\right)^n p_n^\beta(w),$$

and applying the operator T_β to both sides and comparing the coefficients in the two power series we obtain the result. □

1.13. THEOREM. *$MH(I+(i\tan\beta)MH)^{-1}$ is bounded on $L^2(\mu_\beta)$ for $-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$, and its adjoint T_β^* satisfies:*

$$T_\beta^* p_n^\beta(w) = -i\cos^2\beta\, p_{n+1}^\beta(w)$$

and

$$\|T_\beta^* f\| = \cos\beta\|f\| \text{ for each } f \in L^2(\mu_\beta).$$

Also $(1/\cos\beta)T_\beta^*$ *is a right shift of multiplicity one.*

PROOF. The boundedness follows from

$$\|T_\beta p_n^\beta(w)\|/\|p_n^\beta(w)\| = [(\cos\beta)^n/(\cos\beta)^{n-1}] = \cos\beta$$

and from the fact that $\{p_n^\beta(w): n \geq 0\}$ is an orthogonal basis for $L^2(\mu_\beta)$ (by the theorem of Hamburger used in Theorem 1.11). Further $T_\beta^* p_n^\beta = c p_{n+1}^\beta$ for some $c \in \mathbb{C}$ and

$$c \| p_{n+1}^\beta \|^2 = \langle T_\beta^* p_n^\beta, p_{n+1}^\beta \rangle = \langle p_n^\beta, T_\beta p_{n+1}^\beta \rangle = -i \| p_n^\beta \|^2,$$

thus $c = -i \cos^2 \beta$. $\square$

It is symbolically neat to observe that $MH((\cos \beta)I + (i \sin \beta)MH)^{-1}$ is a left shift on $L^2(\mu_\beta)$.

§ 2. THE DIRICHLET PROBLEM FOR $L^2(0, \pi)$

The CR_λ condition was motivated by the Dirichlet problem on the ball in the Heisenberg group H_N (topologically $\cong \mathbb{C}^N \times \mathbb{R}$) for a second-order differential operator L_γ (which is hypoelliptic except for $\gamma = \pm N, \pm(N+2), \pm(N+4), \dots$), solved by Gaveau [4] for $\gamma = 0$. After expressing the problem in terms of harmonic (that is, in the kernel of L_γ) polynomials (see Greiner [5], Dunkl [2], Greiner and Koornwinder [6]) it leads to the following problem:

Given $f \in C[0, \pi]$, find a smooth function g on $\{z \in \mathbb{C} : |z| \leq 1, \ \mathrm{Im}\ z \geq 0\}$ agreeing with f on the half-circle and annihilated by

$$D_{\alpha\beta} := (z - \bar{z}) \frac{\partial^2}{\partial z \partial \bar{z}} - \alpha \frac{\partial}{\partial z} + \beta \frac{\partial}{\partial \bar{z}}$$

in the interior. In the application to H_N, $\alpha = (N - \gamma)/2$, $\beta = (N + \gamma)/2$. The homogeneous polynomial solutions to $D_{\alpha\beta} g = 0$ are the *Heisenberg polynomials*

$$C_n^{(\alpha,\beta)}(z) := \sum_{j=0}^{n} \frac{(\alpha)_j (\beta)_{n-j}}{j!(n-j)!} \bar{z}^j z^{n-j}, \quad n \geq 0.$$

In an effort to discover more about this problem we fix $\mu \in \mathbb{C}$, let $\nu > 0$ and $\alpha := (1 - \mu)\nu$, $\beta := (1 + \mu)\nu$ and let $\nu \to 0$. The operator $D_{\alpha\beta}$ tends to (a multiple of) the ordinary Laplacian off the real diameter, and to $-(1-\mu)\ \partial/\partial z + {} + (1 + \mu)\ \partial/\partial \bar{z}$ on the real diameter. So we get condition CR_λ with $\mu = \mathrm{th}\ \lambda$, whose homogeneous polynomial solutions are

$$e^\lambda z^n + e^{-\lambda} \bar{z}^n = 2 \ \mathrm{ch}\ \lambda \cdot r^n (\cos n\theta + i\ \mathrm{th}\ \lambda \sin n\theta),$$

for $z = re^{i\theta}$. In this section we will use μ or λ, as convenient.

We consider the problem of whether each $f \in L^2(0, \pi)$ can be expanded in the form

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n (\cos n\theta + i\mu \sin n\theta) \text{ with } \sum |a_n|^2 < \infty,$$

(L^2-convergence of the partial sums). If this occurs we will say $\{a_n\}$ *solves* (S_μ) *for* f. To study this problem we identify $L^2(0, \pi)$ with L_E, so we actually consider terms like $\cos n\theta + i\mu(\mathrm{sgn}\ \theta) \sin n\theta$, namely $(I + \mu MH)\cos n\theta$.

2.1. PROPOSITION. *The operator $I + \mu MH$ has dense range in L_E for all $\mu \in \mathbb{C}$; it is one-to-one exactly for $|\mu| \leq 1$, and has a one-dimensional kernel for $|\mu| > 1$.*

PROOF. The statements are easy consequences of MH being a left shift. Thus the range of $I+\mu MH$ always includes all polynomials in w (that is, $\sum_{j=0}^{N} c_j p_j(w)$) and the kernel is easily described as an infinite series in $\{p_n(w)\}$. $\square$

2.2. THEOREM.

1) *If* $|\mu|<1$ *then for each* $f\in L_E$ *there is a unique solution* $\{a_n\}$ *of problem* (S_μ) *for* f, *and* $a_n=\langle (I+\mu MH)^{-1}f, \phi_n\rangle$, *where* $\phi_0:=1$ *and*

$$\phi_n:=2\cos n\theta \text{ for } n\geq 1;$$

2) *if* $|\mu|=1$, *the set of* $f\in L_E$ *for which* (S_μ) *can be solved is a proper dense subspace;*
3) *if* $|\mu|>1$, *problem* (S_μ) *can be solved for each* $f\in L_E$ *with the solution set*

$$a_n=\langle HM(\mu I+HM)^{-1}f, \phi_n\rangle + ci^n q_n\left(\frac{4}{\pi}\lambda_1\right), \quad c\in\mathbb{C},$$

where λ_1 *is the unique number with* $|\mathrm{Im}\,\lambda_1|<\frac{1}{4}\pi$ *such that* $\mathrm{th}\,\lambda_1=1/\mu$.

PROOF. For any $\mu\in\mathbb{C}$, (S_μ) can be solved for f if and only if $f=(I+\mu MH)g$ with $g(\theta)=\sum_{n=0}^{\infty} a_n\cos n\theta$ and $g\in L_E$. Since $I+\mu MH$ has a bounded inverse for $|\mu|<1$ (note $\|MH\|=1$), this establishes (1). By Proposition 2.1 the spectrum of MH is the closed unit disk, but since $I+\mu MH$ is one-to-one for $|\mu|=1$ it must have a proper subspace as range (closed-graph theorem). This shows (2).

For $|\mu|>1$ and $f\in L_E$ let $g=HM(\mu I+HM)^{-1}f$, then $(I+\mu MH)g=$ $=(HM+\mu MH^2M)(\mu I+HM)^{-1}f=f$. The kernel of $I+\mu MH$ is spanned by $\sum_{n=0}^{\infty}(i/\mu)^n p_n(w)$, but we need to expand this as a cosine series. By the generating function (see Appendix) the sum is

$$(\mathrm{ch}\,\lambda_1)\exp(i\lambda_1 w) \text{ where } \mathrm{th}\,\lambda_1=1/\mu \text{ and } |\mathrm{Im}\,\lambda_1|<\tfrac{1}{4}\pi.$$

The boundary value of the analytic function $\exp(i\lambda_1\varrho)$ is $\exp\lambda_1(iw(\theta)-\mathrm{sgn}\,\theta)$, thus the boundary value of the harmonic function

$$\tfrac{1}{2}(\exp(i\lambda_1\varrho(z))+\exp(i\lambda_1\varrho(\bar{z}))) \text{ is } (\mathrm{ch}\,\lambda_1)\exp(i\lambda_1 w).$$

This function has the expansion

$$\tfrac{1}{2}\sum_{n=0}^{\infty}[(iz)^n+(i\bar{z})^n]q_n\left(\frac{4}{\pi}\lambda_1\right)$$

since

$$\exp(i\lambda_1\varrho(z))=\left(\frac{1+z}{1-z}\right)^{2i\lambda_1/\pi}.$$

Thus

$$\sum_{n=0}^{\infty} i^n q_n((4/\pi)\lambda_1)\cos n\theta$$

spans the kernel of $I+\mu MH$, and its norm is $|\mu|/(|\mu|^2-1)^{\frac{1}{2}}$. $\square$

Lauwerier [7] studied some boundary value problems with a particular case amounting to the following: given a function f on $(0,\pi)$, study the function $\sum_{n=1}^{\infty}(a_n/n)r^n \sin n\theta$ for $0<r<1$ where $\{a_n\}$ is determined by

$$f(\theta)=\sum_{n=1}^{\infty} a_n(\cos n\theta+i\mu \sin n\theta),$$

for fixed imaginary μ with $|\mu|>1$. In this work he found the kernel of $I+\mu MH$ and also the biorthogonal set for $\{\cos n\theta+i\mu(\operatorname{sgn}\theta)\sin n\theta : n\geq 1$ for $|\mu|>1$, $n\geq 0$ for $|\mu|<1\}$. He also described the behaviour of the series $\sum_{n=1}^{\infty}(a_n/n)r^n \sin n\theta$ when f satisfies certain smoothness conditions. Here we are mainly concerned with the analysis of the Poisson kernel (which is not in [7]) for the CR_λ problem, and its relation to MH and the group G.

We proceed to the construction of functions satisfying CR_λ. The idea is this: for any cosine series $g=\sum_{n=0}^{\infty} a_n \cos n\theta$ in L_E, the function whose value is

$$\sum_{n=0}^{\infty} a_n(\cos n\theta+i\mu \sin n\theta)=(I+\mu MH)g(\theta) \text{ on } 0<\theta<\pi$$

and is

$$\sum_{n=0}^{\infty} a_n(\cos n\theta-i\mu \sin n|\theta|)=(I-\mu MH)g(|\theta|) \text{ on } -\pi<\theta<0,$$

has (ordinary) Poisson integral

$$\sum_{n=0}^{\infty} a_n r^n(\cos n\theta+i\mu \sin n\theta)(-\pi<\theta\leq\pi,\ 0\leq r<1),$$

which satisfies CR_λ.

2.3. DEFINITION. For $|\mu|<1$, $f\in L^2(0,\pi)$ define $\tilde{f}_\mu\in L^2(T)$ by

$\tilde{f}_\mu(\theta)=f(\theta)$ for $0<\theta<\pi$ and $=(I-\mu MH)(I+\mu MH)^{-1}f(|\theta|)$ for $-\pi<\theta<0$.

2.4. PROPOSITION. *The Poisson integral $P[\tilde{f}_\mu]$ satisfies CR_λ and has f as (L^2) boundary value on $0<\theta<\pi$.*

To be able to extend this result to other L^p spaces we need a more explicit description of the construction of $\tilde{f}_\mu$. We will do this by using the G-action.

For $|\mu|>1$ one may find a solution to the CR_λ problem by piecing together f on $(0,\pi)$ and $(I-\mu MH)HM(\mu I+HM)^{-1}f$ on $(-\pi,0)$ and then taking the Poisson integral. Of course there is a unique (up to scalar multiplication) function $k\in L^2(T)$ so that $P[k]$ satisfies

CR_λ and $k=0$ on $(0,\pi)$,

namely

$$k(\theta)=e^{i\lambda_1 w(\theta)}=|\tan\theta/2|^{-i\pi\lambda_1/2} \text{ on } -\pi<\theta<0,$$

(where $\mathrm{th}\,\lambda_1 = 1/\mu$, see Theorem 2.2). When $\lambda_1 \in \mathbb{R}$, that is $\mu > 1$ or $\mu < -1$, then $k(\theta)$ is bounded on $(-\pi, \pi)$ and continuous except at 0 and $\pm\pi$, thus $P[k]$ is bounded on the open disk.

The Poisson kernel for CR_λ, that is, an integral transform on $(0, \pi)$ which gives the values of $P[\tilde{f}_\mu]$ will be derived by starting at $z = 0$, then using the G-action for real z, $-1 < z < 1$.

We note that the (ordinary) Poisson integral commutes with the action of the Möbius group, hence with the subgroup G. Thus for $f \in L^2(T)$ the Poisson integral satisfies

$$P[f](0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

and

$$P[R_t f](0) = (R_t P[f])(0) = P[f](f_t(0)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (R_t f)(\theta)\, d\theta, \text{ for } t \in \mathbb{R}.$$

It is useful to express this in terms of the adjoint of R_t (see Proposition 1.3).

2.5. PROPOSITION. *For $f \in L_E$, $t \in \mathbb{R}$,*

$$R_t^* f(\arccos(\mathrm{th}\, \tfrac{1}{2}\pi w)) = f(\arccos(\mathrm{th}\, \tfrac{1}{2}\pi(w-t)))(\mathrm{ch}\, \tfrac{1}{2}\pi w)/(\mathrm{ch}\, \tfrac{1}{2}\pi(w-t)),$$

and for $f \in L^2(T)$, $P[f](\mathrm{th}\, \tfrac{1}{4}\pi t) = \langle f, R_t^* 1 \rangle$.

PROOF. It is a simple calculation to find R_t^* using the isomorphism of L_E with $L^2(\mu_0)$. Also for $f \in L^2(T)$, $P[f](0) = \langle f, 1 \rangle$. □

2.6. PROPOSITION. *For $f \in L^2(0, \pi)$, $t \in \mathbb{R}$, $|\mu| < 1$,*

$$P[\tilde{f}_\mu](\mathrm{th}\, \tfrac{1}{4}\pi t) = \langle f, R_t^*(I + \bar{\mu} HM)^{-1} 1 \rangle;$$

and

$$(I + \bar{\mu} HM)^{-1} 1 = (\mathrm{ch}\, \bar{\lambda}) \exp(i\bar{\lambda} w),$$

where

$$\mu = \mathrm{th}\, \lambda \textit{ and } |\mathrm{Im}\, \lambda| < \pi/4.$$

PROOF. First

$$P[\tilde{f}_\mu](\mathrm{th}\, \tfrac{1}{4}\pi t) = R_t P[\tilde{f}_\mu](0) = P[R_t \tilde{f}_\mu](0) = \frac{1}{2\pi} \int_0^{\pi} R_t f(\theta)\, d\theta +$$

$$+ \frac{1}{2\pi} \int_{-\pi}^{0} R_t((I - \mu MH)(I + \mu MH)^{-1}) f(|\theta|)\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{\pi} (I - (I - \mu MH)(I + \mu MH)^{-1}) R_t f(\theta)\, d\theta$$

$$= \frac{1}{\pi} \int_0^{\pi} (I + \mu MH)^{-1} R_t f(\theta)\, d\theta.$$

$$= \langle (I + \mu MH)^{-1} R_t f, 1 \rangle = \langle R_t f, (I + \bar{\mu} HM)^{-1} 1 \rangle,$$

since R_t commutes with the Poisson integral and with MH (see Proposition 1.6).

Further

$$(I+\bar{\mu}HM)^{-1}1 = \sum_{n=0}^{\infty} (-\bar{\mu})^n (HM)^n 1 = \sum_{n=0}^{\infty} (i\bar{\mu})^n p_n(w) = (\text{ch } \bar{\lambda}) \exp (i\bar{\lambda} w)$$

where $\mu = \text{th } \lambda$. $\square$

2.7. COROLLARY. *For* $\lambda \in \mathbb{C}$, $|\text{Im } \lambda| < \pi/4$, $t \in \mathbb{R}$, $f \in L^2(0, \pi)$,

$$P[\tilde{f}_\mu](\text{th } \tfrac{1}{4}\pi t) = \frac{\text{ch } \lambda}{2} \int_{\mathbb{R}} f(\arccos (\text{th } \tfrac{1}{2}\pi w)) \frac{e^{i\lambda(t-w)}}{\text{ch } \frac{1}{2}\pi(t-w)} \, dw,$$

(*where* $\mu = \text{th } \lambda$).

We can expand $P[\tilde{f}_\mu](x)$ in a Taylor series near $x=0$. The coefficients will be realized as integrals of f against the power series expansion of the above integral kernel. Restricting a series $\sum_{n=0}^{\infty} a_n r^n (\cos n\theta + i\mu \sin n\theta)$ to the real diameter gives $\sum_{n=0}^{\infty} a_n x^n$, and so obtaining the coefficient a_n amounts to integrating f against the nth function in a biorthogonal set.

We use renormalized Chebyshev polynomials, namely $t_0 := 1$, $t_n(\cos \theta) = 2 \cos n\theta$, for $n \geq 1$.

2.8. PROPOSITION. *For* $\lambda \in \mathbb{C}$, $w \in \mathbb{R}$, $t \in \mathbb{R}$ *the coefficient of* x^n *in the expansion of*

$$(\text{ch } \lambda) e^{i\lambda(t-w)} \text{ch } \tfrac{1}{2}\pi w / \text{ch } \tfrac{1}{2}\pi(t-w),$$

where $x = \text{th } \frac{1}{4}\pi t$, *is*

$$(\text{ch } \lambda) e^{-i\lambda w} \sum_{j=0}^{n} i^{n-j} q_{n-j}\left(\frac{4}{\pi} \lambda\right) t_j(\text{th } \tfrac{1}{2}\pi w)$$

(*note that* $t_j(\text{th } \frac{1}{2}\pi w(\theta)) = 2 \cos j\theta$ *for* $0 < \theta < \pi$, $j \geq 1$).

PROOF. By the generating function (see Appendix)

$$e^{i\lambda t} = \sum_{n=0}^{\infty} (i \text{ th } \tfrac{1}{4}\pi t)^n q_n\left(\frac{4}{\pi} \lambda\right) = \sum_{n=0}^{\infty} (ix)^n q_n\left(\frac{4}{\pi} \lambda\right).$$

Also

$$\text{ch } \tfrac{1}{2}\pi w / \text{ch } \tfrac{1}{2}\pi(t-w) = (1-x^2)/(1-2x \text{ th } \tfrac{1}{2}\pi w + x^2) = \sum_{n=0}^{\infty} x^n t_n(\text{th } \tfrac{1}{2}\pi w)$$

(the ordinary Poisson kernel on the real diameter for $\cos \theta = \text{th } \frac{1}{2}\pi w$). $\square$

This biorthogonal set was found by Lauwerier [7] in a different way.

The kernel for the whole disk can be quickly obtained from Corollary 2.7 by some simple observations: for $f \in L^2(0, \pi)$ the function $P[\tilde{f}_\mu](\text{th } \frac{1}{4}\pi t)$ has the Taylor series $\sum_{n=0}^{\infty} a_n x^n$ where $x = \text{th } \frac{1}{4}\pi t$ and $f(\theta) = \sum_{n=0}^{\infty} a_n(\cos n\theta + i\mu \sin n\theta)$; in turn the Poisson integral is

$$\sum_{n=0}^{\infty} a_n(2 \text{ ch } \lambda)^{-1}(e^{\lambda} z^n + e^{-\lambda}\bar{z}^n) =$$

$$= (e^{\lambda}/2 \text{ ch } \lambda) \sum a_n z^n + (e^{-\lambda}/2 \text{ ch } \lambda) \sum a_n \bar{z}^n;$$

it remains to allow t to assume complex values in the strip S, mapped by W onto the disk D. The resulting transform will still be a convolution over $\mathbb{R}$ (that is, the image of G), indeed with the function

$$\tfrac{1}{4}\left(\frac{e^{i\lambda t + \lambda}}{\text{ch } \frac{1}{2}\pi t} + \frac{e^{i\lambda \bar{t} - \lambda}}{\text{ch } \frac{1}{2}\pi \bar{t}}\right),$$

$t \in S$. For convolution this is to be interpreted as a function of σ with $t = \sigma + i\tau$ and τ is fixed. Some algebraic manipulation leads to the following:

3.1. DEFINITION. For $\lambda \in \mathbb{C}$, $\sigma \in \mathbb{R}$, $-1 < \tau < 1$ let

$$K_\lambda(\sigma, \tau) := \frac{e^{i\lambda\sigma}}{4|\text{ch } \frac{1}{2}\pi(\sigma + i\tau)|^2} [e^{\pi\sigma/2} \sin ((\tfrac{1}{2}\pi - i\lambda)(1 - \tau)) +$$

$$+ e^{-\pi\sigma/2} \sin ((\tfrac{1}{2}\pi + i\lambda)(1 - \tau))].$$

3.2. PROPOSITION. *For any fixed* $\tau_0 > 0$ (*and* $\tau_0 < 1$)

$$|K_\lambda(\sigma, \tau)| = O(\exp(-(\text{Im } \lambda)\sigma)/\text{ch } \tfrac{1}{2}\pi\sigma)$$

uniformly in $|\tau| \le \tau_0$.

PROOF. Note that $|\text{ch } \frac{1}{2}\pi(\sigma + i\tau)|^2 = \text{sh}^2(\frac{1}{2}\pi\sigma) + \cos^2(\frac{1}{2}\pi\tau)$. Thus for

$$-1 < \tau < 1, \ |K_\lambda(\sigma, \tau)| \le B_\lambda e^{-\sigma \text{ Im } \lambda} \text{ch } \tfrac{1}{2}\pi\sigma/(\text{sh}^2(\tfrac{1}{2}\pi\sigma) + \cos^2(\tfrac{1}{2}\pi\tau))$$

where B_λ is a constant depending only on λ. For

$$-\tau_0 \le \tau \le \tau_0, \ \text{sh}^2(\tfrac{1}{2}\pi\sigma) + \cos^2(\tfrac{1}{2}\pi\tau) \ge \text{sh}^2(\tfrac{1}{2}\pi\sigma) + \cos^2(\tfrac{1}{2}\pi\tau_0) \ge$$
$$\ge \text{ch}^2(\tfrac{1}{2}\pi\sigma)\cos^2(\tfrac{1}{2}\pi\tau_0). \quad \square$$

At this point it becomes convenient to work solely on the strip S. The formulas can be easily converted to the disk by using the various mappings like $z = \text{th } \frac{1}{4}\pi(\sigma + i\tau)$, and $w + i = \varrho(e^{i\theta})$ with $\cos \theta = \text{th } \frac{1}{2}\pi w$, $w \in \mathbb{R}$.

The integrals that we use here all involve limiting behavior at $\pm\infty$ in S (that is, ± 1 in the disk). To suitably control this we introduce the condition BCR_λ (Bounded CR_λ): say f *satisfies* BCR_λ if f satisfies CR_λ on the strip and for each $\tau_0 > 0$ there is a constant A such that $|f(\sigma + i\tau)| \le A \exp(-\sigma \text{ Im } \lambda + \frac{1}{2}\pi|\sigma|)$ for all $\sigma \in \mathbb{R}$, $-\tau_0 \le \tau \le \tau_0$. This is a reasonable restriction as the following shows.

3.3. DEFINITION. For $|\mathrm{Im}\,\lambda|<\frac{1}{2}\pi$, f measurable on $\mathbb{R}+i$ (the upper edge of S) satisfying

$$\int_{\mathbb{R}} |f(w+i)|d\mu_{\mathrm{Im}\,\lambda}(w)<\infty$$

define the Poisson integral of f by

$$P_\lambda[f](\sigma+i\tau)=\int_{\mathbb{R}} f(w+i)K_\lambda(\sigma-w,\tau)dw, \quad (\sigma+i\tau\in S).$$

3.4. PROPOSITION. *For f as in* 3.3, *$P_\lambda[f]$ satisfies BCR_λ.*

PROOF. Since $P_\lambda[f]$ has the form $e^\lambda h+e^{-\lambda}(Jh)$ with h analytic, it must satisfy CR_λ. Also for any $\tau_0>0$

$$\begin{aligned}|P_\lambda[f](\sigma+i\tau)|&\le \int_{\mathbb{R}} |f(w+i)|\,|K_\lambda(\sigma-w,\tau)|dw\\ &\le \int_{\mathbb{R}} |f(w+i)|e^{w\,\mathrm{Im}\,\lambda}(\mathrm{sech}\,\tfrac{1}{2}\pi w)dw\cdot\\ &\quad\cdot \sup_{w\in\mathbb{R}}(e^{-w\,\mathrm{Im}\,\lambda}(\mathrm{ch}\,\tfrac{1}{2}\pi w)|K_\lambda(\sigma-w,\tau)|), \text{ for } -\tau_0\le\tau\le\tau_0.\end{aligned}$$

The supremum is bounded by

$$\begin{aligned}&\sup_w(Be^{-w\,\mathrm{Im}\,\lambda}\mathrm{ch}\,\tfrac{1}{2}\pi w\; e^{-(\sigma-w)\mathrm{Im}\,\lambda}/\mathrm{ch}\,\tfrac{1}{2}\pi(\sigma-w))\le\\ &\le B\exp(-\sigma\,\mathrm{Im}\,\lambda+\tfrac{1}{2}\pi|\sigma|)\end{aligned}$$

for some constant B, by Proposition 3.2. □

It is important that K_λ acts as reproducing kernel for the class BCR_λ. We adjust K_λ for narrower strips.

3.5. THEOREM. *If $|\mathrm{Im}\,\lambda|<\frac{1}{2}\pi$ and f satisfies BCR_λ on S then for each τ with $0<\tau<1$,*

$$f(\sigma+iu)=(1/\tau)\int_{\mathbb{R}} f(\sigma-w+i\tau)K_\lambda\left(\frac{w}{\tau},\frac{u}{\tau}\right)dw \textit{ for } -\tau<u<\tau,\ \sigma\in\mathbb{R}.$$

PROOF. If $\lambda=0$ this is the ordinary Poisson kernel for the strip (see Widder [10]) applied to even functions. Now assume $\lambda\neq0$ and f satisfies BCR_λ. Then $f=e^\lambda g+e^{-\lambda}Jg$ for a unique analytic function g on S, and g can be determined from

$$g(\sigma+i\tau)=(2\,\mathrm{sh}\,2\lambda)^{-1}(e^\lambda f(\sigma+i\tau)-e^{-\lambda}f(\sigma-i\tau)).$$

Passing to the consideration of analytic function we state a version of the Cauchy integral formula for the strip (it can be proved by the calculus of residues):

$$h(\sigma+iu)=(1/4\tau)\int_{\mathbb{R}} [h(w+i\tau)+h(w-i\tau)]\ \mathrm{sech}\,\frac{\pi}{2\tau}\,[(\sigma-w)+iu]dw,$$

valid for $\sigma \in \mathbb{R}$, $-\tau < u < \tau < 1$, provided $|h(\sigma+iu)| < Ae^{a|\sigma|}$ for some $a < \pi/2\tau$ and $A < \infty$ in the closed strip $|u| \le \tau$.

Let $h(\xi) := e^{-i\lambda\xi/\tau} g(\xi)$ for $\xi \in S$, (and g from above). Then h satisfies the hypotheses for the formula since $|g(\sigma+iu)| = O(\exp(-\sigma \text{ Im } \lambda + \frac{1}{2}\pi|\sigma|))$ uniformly in $-\tau \le u \le \tau$ and $|(\text{Im } \lambda)(1/\tau - 1)| + \frac{1}{2}\pi < \frac{1}{2}\pi/\tau$ for $|\text{Im } \lambda| < \frac{1}{2}\pi$ and $0 < \tau < 1$. This gives the formula

$$g(\sigma+iu) = \frac{e^{i\lambda(\sigma+iu)/\tau}}{4\tau} \int_{\mathbb{R}} e^{-i\lambda w/\tau}[e^{\lambda} g(w+i\tau) +$$

$$+ e^{-\lambda} g(w-i\tau)] \operatorname{sech} \frac{\pi}{2\tau}(\sigma - w + iu) dw$$

$$= \frac{e^{-\lambda u/\tau}}{4\tau} \int_{\mathbb{R}} e^{i\lambda(\sigma-w)/\tau} \operatorname{sech} \frac{\pi}{2\tau}(\sigma - w + iu) f(w+i\tau) dw.$$

Now use this formula for $g(\sigma - iu)$ and combine to obtain

$$f(\sigma+iu) = e^{\lambda} g(\sigma+iu) + e^{-\lambda} g(\sigma-iu) = (1/\tau) \int_{\mathbb{R}} f(\sigma - w + i\tau) K_{\lambda}(w/\tau, u/\tau) dw.$$

□

3.6. COROLLARY. *If g satisfies BCR_{λ} on S ($|\text{Im } \lambda| < \frac{1}{2}\pi$) and*

$$(1/\tau) \int_{\mathbb{R}} |g(\sigma+i\tau)| e^{(\text{Im } \lambda)\sigma/\tau} \operatorname{sech} \frac{\pi\sigma}{2\tau} d\sigma \to 0$$

as $\tau \to 1-$, then $g \equiv 0$.

PROOF. Choose numbers $0 < \tau_0 < \tau_1 < 1$ and then use the formula for the strip $-\tau_0 \le u \le \tau_0$ with τ restricted by $\tau_1 \le \tau < 1$. By an integral inequality and the bound from Proposition 3.2 we obtain

$$|g(\sigma+iu)| < A \exp((-\sigma \text{ Im } \lambda + \tfrac{1}{2}\pi|\sigma|)/\tau) \cdot$$

$$\cdot (1/\tau) \int_{\mathbb{R}} |g(w+i\tau)| e^{(\text{Im } \lambda)w/\tau} \operatorname{sech} \frac{\pi w}{2\tau} dw.$$

Now fix σ and u, and let $\tau \to 1$. □

3.7. PROPOSITION. *If $-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$, and $1 > \tau > (2|\beta| - \pi)/(2|\beta| + \pi)$ then $K_{i\beta}(\sigma, \tau) > 0$. Further if $\lambda = \alpha + i\beta$, ($\alpha \in \mathbb{R}$) then there is a constant A_{λ} such that $|K_{\lambda}(\sigma, \tau)| < A_{\lambda} K_{i\beta}(\sigma, \tau)$ for $\sigma \in \mathbb{R}$, $0 \le \tau < 1$.*

PROOF. From Definition 3.1 we see that we are concerned with the positivity of $\sin((\frac{1}{2}\pi \pm \beta)(1-\tau))$. The argument is in $(0, \pi)$ if τ satisfies the stated bounds.

Next let $\lambda = \alpha + i\beta$. Some calculation leads to

$$(|K_{\lambda}(\sigma,\tau)|/K_{i\beta}(\sigma,\tau))^2 = 1 + \operatorname{sh}^2(\alpha(1-\tau))(1 - \cos^2(\tfrac{1}{2}\pi\tau)/\operatorname{ch}^2(\tfrac{1}{2}\pi\sigma))/$$

$$(\sin((1-\tau)\tfrac{1}{2}\pi)\cos((1-\tau)\beta) + \operatorname{th} \tfrac{1}{2}\pi\sigma \cos((1-\tau)\tfrac{1}{2}\pi)\sin((1-\tau)\beta))^2.$$

The denominator always lies between

$$\sin^2((\tfrac{1}{2}\pi - \beta)(1-\tau)) \text{ and } \sin^2((\tfrac{1}{2}\pi + \beta)(1-\tau)),$$

and the numerator is bounded by 1. Thus the fraction is bounded by $(\mathrm{sh}\ (\alpha x)/\sin\ (bx))^2$ where $b=\frac{1}{2}\pi\pm\beta$, and $x=1-\tau$. But this is an increasing function on $0<x<\pi/|b|$. Hence the original ratio $(|K_\lambda|/K_{i\beta})^2$ is bounded by $1+\mathrm{sh}^2\ \alpha/\sin^2\ (\frac{1}{2}\pi-|\beta|)$ for $0\leq\tau<1$. □

In fact, as one can see from the proof, for any τ_0 with

$$0>\tau_0>(2|\beta|-\pi)/(2|\beta|+\pi)$$

there is a constant A (depending on λ and τ_0) such that $|K_\lambda(\sigma,\tau)|<AK_{i\beta}(\sigma,\tau)$ for $1>\tau\geq\tau_0$. This will not be needed in the further development.

3.8. THEOREM. *For $\lambda\in\mathbb{C}$, $|\mathrm{Im}\ \lambda|<\frac{1}{2}\pi, f\in C[0,\pi]$ the Poisson integral $P_\lambda[f]$ extends to a continuous function on $\{z\in\mathbb{C}\colon |z|\leq 1,\ \mathrm{Im}\ z\geq 0\}$ agreeing with f on the upper half-circle.*

PROOF. We consider f as a function on $\mathbb{R}+i$ which is continuous and has limits at $\pm\infty$, by means of the map w. Thus f is uniformly continuous, and so it suffices to show: for any $\varepsilon>0$ there exists $M>0$ and $\tau_0<1$ such that:

$$(1)\qquad \begin{cases} |P_\lambda[f](\sigma+i\tau)-f(\sigma+i)|<\varepsilon \\ \text{for } \tau_0<\tau<1, \text{ all } \sigma\in\mathbb{R}; \end{cases}$$

$$(2)\qquad \begin{cases} |P_\lambda[f](\sigma+i\tau)-f(\pm\infty+i)|<\varepsilon \\ \text{for } \sigma>M \text{ (for } +\infty) \text{ or } \sigma<-M \text{ (for } -\infty) \text{ and all } \tau \text{ with } 0\leq\tau<1. \end{cases}$$

Part (1) follows from the standard convolution approximate identity argument and the facts

$$\int_{\mathbb{R}} K_\lambda(\sigma,\tau)d\sigma=1,\quad \int_{\mathbb{R}} |K_\lambda(\sigma,\tau)|d\sigma=A_\lambda,\quad (0\leq\tau<1),$$

and

$$\int_{|\sigma|>\delta} |K_\lambda(\sigma,\tau)|d\sigma\to 0 \text{ as } \tau\to 1 \text{ for each } \delta>0.$$

The first is just $P_\lambda[1]$, and the second and third follow from the same claims for $K_{i\beta}(\sigma,\tau)$ and Proposition 3.7, where $\beta=\mathrm{Im}\ \lambda$. But

$$\int_{|\sigma|>\delta} K_{i\beta}(\sigma,\tau)d\sigma\leq\max\ [\sin\ ((\tfrac{1}{2}\pi+\beta)(1-\tau)),\ \sin\ ((\tfrac{1}{2}\pi-\beta)(1-\tau))]\cdot$$

$$\cdot\int_{|\sigma|>\delta} (e^{-\beta\sigma}\mathrm{ch}\ \tfrac{1}{2}\pi\sigma)(\mathrm{sh}^2\ \tfrac{1}{2}\pi\sigma)^{-1}d\sigma,$$

which tends to 0 as $\tau\to 1$, for each fixed $\delta>0$.

To prove (2) we first observe that

$$\lim_{M\to\infty}\int_{|\sigma|>M} |K_\lambda(\sigma,\tau)|d\sigma=0 \text{ uniformly in } \tau \text{ in } 0\leq\tau<1,$$

since

$$\int_{|\sigma|>M} |K_\lambda(\sigma,\tau)|d\sigma<A_\lambda\int_{|\sigma|>M} (e^{-\beta\sigma}\mathrm{ch}\ \tfrac{1}{2}\pi\sigma)(\mathrm{sh}^2\ \tfrac{1}{2}\pi\sigma)^{-1}d\sigma.$$

Now given $\varepsilon>0$ choose M so that $|f(\sigma+i)-f(\infty+i)|<\varepsilon$ for $\sigma>\frac{1}{2}M$ and

$$\int_{|\sigma|>\frac{1}{2}M} |K_\lambda(\sigma,\tau)|d\sigma<\varepsilon \text{ for } 0\le\tau<1.$$

If $\sigma>M$ then

$$P_\lambda[f](\sigma+i\tau)-f(\infty+i)=$$
$$=\Big(\int_{w>\frac{1}{2}M}+\int_{w<\frac{1}{2}M}\Big)K_\lambda(\sigma-w,\tau)(f(w+i)-f(\infty+i))dw.$$

The first integral is bounded by $A_\lambda\varepsilon$, and the second by

$$2\|f\|_\infty \int_{w<\frac{1}{2}M} |K_\lambda(\sigma-w,\tau)|dw=2\|f\|_\infty \int_{w>\sigma-\frac{1}{2}M} |K_\lambda(w,\tau)|dw;$$

by assumption $\sigma>M$ and so $\sigma-\frac{1}{2}M>\frac{1}{2}M$ and we get the bound $2\|f\|_\infty\varepsilon$. □

The continuous function result together with the following identities will be used to study L^p-behavior.

3.9. PROPOSITION. *For $\lambda\in\mathbb{C}$, $|\mathrm{Im}\ \lambda|<\frac{1}{2}\pi$*

(1) $$\frac{1}{2}\int_{\mathbb{R}} e^{i\lambda w}\operatorname{sech}\tfrac{1}{2}\pi w\, dw=\operatorname{sech}\lambda;$$

(2) $$\begin{cases} \displaystyle\int_{\mathbb{R}} e^{i\alpha w}K_\lambda(\sigma-w,\tau)dw=e^{i\alpha\sigma}\frac{\operatorname{ch}(\lambda-\alpha\tau)}{\operatorname{ch}(\lambda-\alpha)} \\ \textit{for } \sigma\in\mathbb{R},\ -1<\tau<1 \textit{ and } |\mathrm{Im}\ (\lambda-\alpha)|<\frac{1}{2}\pi; \end{cases}$$

(3) $$\begin{cases} \displaystyle K_\lambda(\sigma,u)=(1/\tau)\int_{\mathbb{R}} K_\lambda\Big(\frac{w}{\tau},\frac{u}{\tau}\Big)K_\lambda(\sigma-w,\tau)dw \\ \textit{for } \sigma\in\mathbb{R},\ -\tau<u<\tau<1; \end{cases}$$

(4) $$\begin{cases} \displaystyle (1/\tau)\int_{\mathbb{R}} e^{i\lambda w/\tau}K_\lambda(\sigma-w,\tau)\operatorname{sech}\frac{\pi w}{2\tau}\,dw=e^{i\lambda\sigma}\operatorname{sech}\tfrac{1}{2}\pi\sigma, \\ \textit{for } \sigma\in\mathbb{R},\ -1<\tau<1. \end{cases}$$

PROOF. The first integral is done by use of the contour which is the positively oriented boundary of the rectangle with vertices $\pm A$, $\pm A+2i$, for some $A>0$. The integrand has just one pole at $w=i$. Also the integral from $A+2i$ to $-A+2i$ is $e^{-2\lambda}$ times the integral from $-A$ to A. The other two pieces $\to 0$ as $A\to\infty$.

For (2) start with the analytic function $g(\sigma+i\tau)=(2\ \operatorname{ch}(\lambda-\alpha))^{-1}e^{i\alpha(\sigma+i\tau)}$ and let $f=e^\lambda g+e^{-\lambda}Jg$. Then f satisfies BCR_λ and has the value $e^{i\alpha\sigma}$ on $\tau=1$. The argument of Theorem 3.5 can be almost directly used to show that K_λ reproduces f from its boundary values. Further

$$f(\sigma+i\tau)=\operatorname{ch}(\lambda-\alpha\tau)/\operatorname{ch}(\lambda-\alpha).$$

For (3) we again use Theorem 3.5, this time applied to $f(\sigma+i\tau):=K_\lambda(\sigma,\tau)$. Finally (4) is the special case of (3) with $u=0$. □

We proceed to the L^p aspects of the Poisson kernel and its boundary behaviour. Clearly any $f\in L^p(\mu_{\operatorname{Im}\lambda})$, $1\le p\le\infty$ has a Poisson integral (see Definition 3.3). However for $L^p(0,\pi)$ there is a bound for $\operatorname{Im}\lambda$ depending on p.

3.10. PROPOSITION. *If* $1<p\le\infty$ *and* $|\beta|<\frac{1}{2}\pi(1-1/p)$ *then* $L^p(0,\pi)\subset L^1(\mu_\beta)$.

PROOF. By Hölder's inequality

$$\int_{\mathbb{R}} |f(w+i)|e^{\beta w}\operatorname{sech}\tfrac{1}{2}\pi w\,dw<\infty \text{ for all } f\in L^p(\mu_0)$$

if and only if

$$\int_{\mathbb{R}} e^{\beta q w}\operatorname{sech}\tfrac{1}{2}\pi w\,dw<\infty, \text{ that is, } |\beta|<\pi/(2q)$$

where $1/p+1/q=1$. □

From Theorem 3.5 one might expect the measure that appears in the reproducing formula 3.9(4), namely $(1/\tau)\operatorname{sech}\pi w/2\tau\,dw$ on the lines $\tau=$constant in S. However the factor $e^{w\operatorname{Im}\lambda}$ in K_λ can not be ignored, as we will see.

3.11. THEOREM. (1) *Let* $1\le p<\infty$, $|\operatorname{Im}\lambda|<\frac{1}{2}\pi(1-1/p)$ *for* $p>1$ *or* $\operatorname{Im}\lambda=0$ *for* $p=1$, *and* $0<\tau<1$ *then for each* $f\in L^p(\mu_0)$ $(\cong L^p(0,\pi))$

$$(1/2\tau)\int_{\mathbb{R}} |P_\lambda[f](\sigma+i\tau)|^p \exp\left(\sigma\operatorname{Im}\lambda(1-\tau)/\tau\right)\left(\operatorname{sech}\frac{\pi\sigma}{2\tau}\right)d\sigma$$

$$\le A_\lambda^p\left(\frac{\cos((1+\tau(q-1))\beta)}{\cos\beta q}\right)^{p-1}\int_{\mathbb{R}} |f(\sigma+i)|^p d\mu_0(\sigma),$$

(*where* $\beta=\operatorname{Im}\lambda$, $1/p+1/q=1$);

(2) *Let* $1\le p<\infty$, $|\operatorname{Im}\lambda|<\frac{1}{2}\pi$, $0<\tau<1$ *then for each* $f\in L^p(\mu_\beta)$ (*where* $\beta=\operatorname{Im}\lambda$),

$$\int_{\mathbb{R}} |P_\lambda[f](\sigma\tau+i\tau)|^p d\mu_\beta(\sigma)\le A_\lambda^p\int_{\mathbb{R}} |f(\sigma+i)|^p d\mu_\beta(\sigma).$$

PROOF. Both parts can be proved at once. We consider only bounded continuous functions f with limits at $\pm\infty$ (that is, $C[0,\pi]$) and only $1<p<\infty$; the case $p=1$ is an obvious modification.

Let B be a real constant to be chosen later, satisfying $|\beta-Bq/p|<\frac{1}{2}\pi$, then

$$|P_\lambda[f](\sigma+i\tau)|=\left|\int_{\mathbb{R}} f(w+i)K_\lambda(\sigma-w,\tau)dw\right|$$

$$\le A_\lambda\int_{\mathbb{R}} |f(w+i)|K_{i\beta}(\sigma-w,\tau)dw$$

(see Proposition 3.7)

$$\leq A_\lambda \Big(\int_{\mathbb{R}} |f(w+i)|^p e^{Bw} K_{i\beta}(\sigma - w, \tau) dw \Big)^{1/p}$$

$$\Big(\int_{\mathbb{R}} e^{-Bwq/p} K_{i\beta}(\sigma - w, \tau) dw \Big)^{1/q} =$$

$$= A_\lambda e^{-B\sigma/p} \left(\frac{\cos(\beta - Bq\tau/p)}{\cos(\beta - Bq/p)} \right)^{1/q} \cdot$$

$$\cdot \Big(\int_{\mathbb{R}} |f(w+i)|^p e^{Bw} K_{i\beta}(\sigma - w, \tau) dw \Big)^{1/p}.$$

The integral was computed using Proposition 3.9(2). Take the final inequality for $|\dot{P}_\lambda[f](\sigma + i\tau)|$, raise both sides to the pth power (note $p/q = p-1$), multiply both sides by

$$\exp\left(\left(\frac{\beta}{\tau} + B\right)\sigma\right) \Big/ \left(2\tau \operatorname{ch} \frac{\pi\sigma}{2\tau}\right)$$

and integrate over $\sigma \in \mathbb{R}$. The result is

$$(1/2\tau) \int_{\mathbb{R}} |P_\lambda[f](\sigma + i\tau)|^p \exp\left(\left(\frac{\beta}{\tau} + B\right)\sigma\right) \operatorname{sech} \frac{\pi\sigma}{2\tau} \, d\sigma$$

$$\leq A_\lambda^p \left(\frac{\cos(\beta - Bq\tau/p)}{\cos(\beta - Bq/p)} \right)^{p-1} \tfrac{1}{2} \int_{\mathbb{R}} |f(w+i)|^p e^{Bw} dw \cdot$$

$$\cdot \frac{1}{\tau} \int_{\mathbb{R}} e^{\beta\sigma/\tau} K_{i\beta}(\sigma - w, \tau) \operatorname{sech} \frac{\pi\sigma}{2\tau} \, d\sigma.$$

The latter integral is $e^{\beta w} \operatorname{sech} \frac{1}{2}\pi w$ by Proposition 3.9(4). Choose $B = -\beta$ to get (1) and $B = 0$ to get (2). □

We can now establish a boundary value result; recall the function

$$F_{\tau,\lambda}(\sigma) = \exp(\operatorname{Im} \lambda (1/\tau - 1)\sigma) \operatorname{ch} \tfrac{1}{2}\pi\sigma \Big/ \left(\tau \operatorname{ch} \frac{\pi\sigma}{2\tau}\right)$$

from Theorem 3 in the introduction.

3.12. THEOREM. *Let* $1 \leq p < \infty$ *and* $\lambda \in \mathbb{C}$ *with* $|\operatorname{Im} \lambda| < \frac{1}{2}\pi(1 - 1/p)$ *for* $p > 1$ *or* $\operatorname{Im} \lambda = 0$ *for* $p = 1$ *and let* $f \in L^p(\mu_0)$ $(\cong L^p(0, \pi))$, *then*

$$\int_{\mathbb{R}} |f(\sigma + i) - P_\lambda[f](\sigma + i\tau) F_{\tau,\lambda}(\sigma)^{1/p}|^p d\mu_0(\sigma) \to 0 \text{ as } \tau \to 1-.$$

PROOF. Let $\varepsilon > 0$ then there exists $g \in C[0, \pi]$ (moved to $\mathbb{R} + i$) such that $\|f - g\|_p < \varepsilon$. By Theorem 3.11,

$$\int_{\mathbb{R}} |P_\lambda[f](\sigma + i\tau) - P_\lambda[g](\sigma + i\tau)|^p F_{\tau,\lambda}(\sigma) d\mu_0(\sigma) < A_\lambda^p \varepsilon^p.$$

Thus

$$\|f-P_\lambda[f]F_{\tau,\lambda}^{1/p}\|_p\leq\|f-g\|_p+\|g-P_\lambda[g]F_{\tau,\lambda}^{1/p}\|_p+\|P_\lambda[f]F_{\tau,\lambda}^{1/p}-P_\lambda[g]F_{\tau,\lambda}^{1/p}\|_p.$$

But $P_\lambda[g]$ converges uniformly to g as $\tau\to 1$ (by Theorem 3.8) and $0<F_{\tau,\lambda}(\sigma)<<2/\tau$ so by the dominated convergence theorem $\|g-P_\lambda[g]F_{\tau,\lambda}^{1/p}\|_p\to 0$ as $\tau\to 1$. The bound for $F_{\tau,\lambda}$ comes easily from the bound ch $\frac{1}{2}\pi\sigma/\text{ch}\ (\pi\sigma/2\tau)\leq\leq 2\exp\frac{1}{2}\pi(1-1/\tau)|\sigma|$. □

By a similar argument we could show that if $f\in L^p(\mu_\beta)$ with $1\leq p<\infty$, and $\lambda=\alpha+i\beta$, with $|\beta|<\frac{1}{2}\pi$, then

$$\int_{\mathbb{R}}|f(\sigma+i)-P_\lambda[f](\sigma+i\tau)F_{\tau,\lambda}^{1/p}(\sigma)|^p d\mu_\beta(\sigma)\to 0\text{ as }\tau\to 1-.$$

3.13. PROPOSITION. *Under the hypotheses of Theorem* 3.12, $P_\lambda[f]$ *is the unique function satisfying* BCR_λ *and having* f *as boundary value in the sense of the Theorem.*

PROOF. If g is a function satisfying BCR_λ and converging to f as in 3.12 then

$$(1/\tau)\int_{\mathbb{R}}|g(\sigma+i\tau)-P_\lambda[f](\sigma+i\tau)|e^{(\sigma\,\mathrm{Im}\,\lambda)/\tau}\mathrm{sech}\,\frac{\pi\sigma}{2\tau}\,d\sigma$$

$$\leq(A/\tau)\left[\int_{\mathbb{R}}|g(\sigma+i\tau)-P_\lambda[f](\sigma+i\tau)|^p\exp(\sigma\,\mathrm{Im}\,\lambda(1/\tau-1))\cdot\right.$$

$$\left.\cdot\,\mathrm{sech}\,\frac{\pi\sigma}{2\tau}\,d\sigma\right]^{1/p}$$

for some constant A by Hölder's inequality, and the right side tends to 0 as $\tau\to 1-$. (Roughly $A=O([\cos(q\,\mathrm{Im}\,\lambda)]^{-1/q})$, where $1/p+1/q=1$.) By Corollary 3.6, $g-P_\lambda[f]\equiv 0$. □

We note that if $\frac{1}{2}\pi(1-1/p)<|\mathrm{Im}\,\lambda|<\frac{1}{2}\pi$ then there exists a nonzero function f satisfying BCR_λ such that

$$\lim_{\tau\to 1-}\int_{\mathbb{R}}|f(\sigma+i\tau)|^p e^{\sigma(\mathrm{Im}\,\lambda)(1/\tau-1)}\mathrm{sech}\,\frac{\pi\sigma}{2\tau}\,d\sigma=0,$$

namely $f(\sigma+i\tau)=\mathrm{ch}\,(\lambda(1-\tau)+\varepsilon i\pi\tau/2)\exp(i\sigma(\lambda-i\varepsilon\pi/2))$ with $\varepsilon=\mathrm{sgn}\,(\mathrm{Im}\,\lambda)$; this is to get the exponent in the correct interval.

The case $\lambda\in\mathbb{R}$ allows a somewhat nicer boundary value result, namely Theorem 2 in the introduction. By Proposition 3.7, $|K_\lambda(\sigma,\tau)|<A_\lambda K_0(\sigma,\tau)$, but K_0 is the ordinary Poisson kernel for even harmonic functions. As a result, by an argument similar to that of Theorem 3.11, one can show that

$$|P_\lambda[f](\sigma+i\tau)|^p\leq A_\lambda^p P_0[|f|^p](\sigma+i\tau)$$

for $f\in L^p(0,\pi)$. The latter is the even harmonic majorant of $|f|^p$. Thus Theorem 2 follows from the standard results for the ordinary Poisson kernel.

Finally as a nontrivial example we find the CR_λ Poisson integral for $f(\theta) = \sin\theta$ on $0<\theta<\pi$, and the corresponding Fourier series. Again the Pollaczek polynomials appear.

3.14. PROPOSITION. *Let* $f(\theta)=\sin\theta$ *on* $0<\theta<\pi$ *and* $\lambda\in\mathbb{C}$, $|\mathrm{Im}\,\lambda|<\frac{1}{2}\pi$, *then*

$$P_\lambda[f](\sigma+i\tau)=\frac{e^{i\lambda\sigma/2}}{\mathrm{sh}\,\lambda}\left[e^{\lambda(1-\tau/2)}\frac{\sin\frac{1}{2}\lambda(\sigma+i\tau)}{\mathrm{sh}\,\frac{1}{2}\pi(\sigma+i\tau)}+\right.$$

$$\left.+e^{-\lambda(1-\tau/2)}\frac{\sin\frac{1}{2}\lambda(\sigma-i\tau)}{\mathrm{sh}\,\frac{1}{2}\pi(\sigma-i\tau)}\right],\quad \sigma+i\tau\in S$$

and (*with variable now moved to the disk*)

$$P_\lambda[f](re^{i\theta})=2\lambda/(\pi\ \mathrm{th}\ \lambda)+(4i\lambda^2/(\pi^2\ \mathrm{th}\ \lambda))r(\cos\theta+i(\mathrm{th}\ \lambda)\sin\theta)+$$

$$+\frac{8\lambda(4\lambda^2+\pi^2)}{\pi^3\ \mathrm{th}\ \lambda}\sum_{n=2}^{\infty}\frac{i^n}{n(n^2-1)}p_{n-2}\left(\frac{4}{\pi}\lambda;4,0\right)r^n(\cos n\theta+i(\mathrm{th}\ \lambda)\sin n\theta),$$

$$0\le r<1,\quad -\pi\le\theta\le\pi.$$

PROOF. Observe $\sin\theta=\mathrm{sech}\ \frac{1}{2}\pi w(\theta)$. First we find the analytic part of $P_\lambda[f]$, that is,

$$g(\sigma+i\tau):=\frac{1}{4}\int_{\mathbb{R}}\frac{e^{i\lambda(w+i\tau)}}{\mathrm{ch}\ \frac{1}{2}\pi(w+i\tau)}\ \frac{dw}{\mathrm{ch}\ \frac{1}{2}\pi(\sigma-w)}$$

$$=\frac{e^{i\lambda(\sigma+i\tau)/2}}{\mathrm{sh}\ \lambda}\ \frac{\sin\frac{1}{2}\lambda(\sigma+i\tau)}{\mathrm{sh}\ \frac{1}{2}\pi(\sigma+i\tau)},\quad -1<\tau<1$$

(see Theorem 3.5). The integral is done by residue calculus using the rectangle with vertices $\pm A$, $\pm A+2i$ (for $A>|\sigma|$) as contour. The integrand has poles at $\sigma+i$ and $i(1-\tau)$ inside the contour.

Next $P_\lambda[f](\sigma+i\tau)=e^\lambda g(\sigma+i\tau)+e^{-\lambda}g(\sigma-i\tau)$. The coefficient in the Fourier series of $P_\lambda[f](re^{i\theta})$ of the term $r^n(\cos n\theta+i(\mathrm{th}\ \lambda)\sin n\theta)$ is exactly $2\mathrm{ch}\ \lambda$ times the coefficient of z^n in

$$g(\varrho(z))=\frac{1}{4iz\ \mathrm{sh}\ \lambda}\ [(1-z)^{1-2i\lambda/\pi}(1+z)^{1+2i\lambda/\pi}-(1-z^2)]$$

(a simple consequence of $\varrho(z)=(2/\pi)\log((1+z)/(1-z))$. The coefficients are found in terms of the polynomials $p_n(4\lambda/\pi;-2,0)$, then transformed by a formula from the Appendix. □

3.15. COROLLARY. *On the real diameter of D,*

$$P_\lambda[f](\mathrm{th}\ \tfrac{1}{4}\pi t)=2\,\frac{e^{i\lambda t/2}}{\mathrm{th}\ \lambda}\ \frac{\sin\frac{1}{2}\lambda t}{\mathrm{sh}\ \frac{1}{2}\pi t},$$

and in particular $P_\lambda[f](0)=2\lambda/(\pi\ \mathrm{th}\ \lambda)$. *Further the boundary value of* $P_\lambda[f]$ *on the lower half-circle* $(-\pi<\theta<0)$ *is given by*

$$-\sin\theta(2\ \mathrm{ch}\ \lambda|\tan\theta/2|^{-2i\lambda/\pi}-1).$$

These formulas illustrate some of the behaviour of P_λ for various λ. Notably, $P_\lambda[f](0)$ diverges as $\lambda \to \pm\infty$ (that is, th $\lambda \to \pm 1$).

In summary, this paper illustrates some interesting connections between the Hilbert transform and Pollaczek polynomials and also makes some suggestions for one's intuition regarding the Dirichlet problem on the Heisenberg group.

APPENDIX

This is a collection of notations and results dealing with a family of *Pollaczek polynomials*. They have been adapted from Pollaczek's paper [8] (see also Chihara [1], p. 179 ff., p. 186 ff.).

For parameters β, γ with $-\pi/2<\beta<\pi/2$ and $\gamma \in \mathbb{R}$ there is a family of polynomials defined by

$$(1-ie^{i\beta}t/\cos\beta)^{(ix-\gamma)/2}(1+ie^{-i\beta}t/\cos\beta)^{-(ix+\gamma)/2}$$

$$=\sum_{n=0}^{\infty} t^n p_n(x;\gamma,\beta), \quad |t|<1.$$

This is equivalent to

$$\left(\frac{\operatorname{ch}(\alpha-i\beta)}{\cos\beta}\right)^{\gamma} e^{i\alpha x} = \sum_{n=0}^{\infty}\left(\frac{i \operatorname{th}\alpha}{1-i\operatorname{th}\alpha\tan\beta}\right)^{n} p_n(x;\gamma,\beta),$$

(valid at least for $|\operatorname{th}\alpha\tan\beta|<1$).

Explicitly

$$p_n(x;\gamma,\beta)=\left(\frac{ie^{i\beta}}{\cos\beta}\right)^{n}\sum_{j=0}^{n}\frac{((\gamma+ix)/2)_j}{j!}\frac{((\gamma-ix)/2)_{n-j}}{(n-j)!}(-1)^j e^{-2ji\beta}$$

and

$$p_n(x;\gamma,\beta)=(\cos\beta)^{-n}P_n^{(\gamma/2)}(\tfrac{1}{2}x;\beta+\tfrac{1}{2}\pi)$$

in Pollaczek's notation [8].

The polynomials satisfy a three-term recurrence

$$p_{n+1}(x;\gamma,\beta)=(1/(n+1))[(x-(2n+\gamma)\tan\beta)p_n(x;\gamma,\beta)$$

$$-(n+\gamma-1)(\sec^2\beta)p_{n-1}(x;\gamma,\beta)]$$

(thus p_n has leading coefficient $1/n!$, and is a real polynomial).

For $\gamma>0$ there is an orthogonality on $\mathbb{R}$,

$$\frac{(2\cos\beta)^{\gamma}}{4\pi\Gamma(\gamma)}\int_{\mathbb{R}} p_n(x;\gamma,\beta)p_m(x;\gamma,\beta)e^{\beta x}|\Gamma((\gamma+ix)/2)|^2dx$$

$$=\delta_{mn}(\gamma)_n/(n!(\cos\beta)^{2n}).$$

In this paper we only need this for $\gamma=1$ when the weight is $\pi e^{\beta x}\operatorname{sech}\frac{1}{2}\pi x$.

The following deal with the shift and Sheffer polynomial structure (see Rota [9], Ch. 2):

$$p_n(x_1+x_2;\gamma_1+\gamma_2,\beta)=\sum_{j=0}^{n} p_j(x_1;\gamma_1,\beta)p_{n-j}(x_2;\gamma_2,\beta)$$

(in particular, the case $\gamma_1=\gamma_2=0$ gives a binomial family);

$$p_n(x;\gamma+1,\beta)=(2\cos\beta)^{-1}(e^{-i\beta}p_n(x+i;\gamma,\beta)$$
$$+e^{i\beta}p_n(x-i;\gamma,\beta)), \text{ that is, for } x\in\mathbb{R},$$
$$p_n(x;\gamma+1,\beta)=\mathrm{Re}(e^{-i\beta}p_n(x+i;\gamma,\beta))/\cos\beta.$$

For brevity we adapt the notations:

$$p_n(x):=p_n(x;1,0);\ q_n(x):=p_n(x;0,0);$$
$$p_n^\beta(x):=p_n(x;1,\beta).$$

The seemingly degenerate values $\gamma=1-N$ for $N=1,2,3,\ldots$ are not so bad because

$$p_n(x;1-N,0)=\frac{(2i)^N}{(-n)_N}((1-N+ix)/2)_N p_{n-N}(x;N+1,0) \text{ for } n\geq N.$$

For example

$$p_n(x;0,0)=(x/n)p_{n-1}(x;2,0),\ n\geq 1.$$

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ADDED IN PROOF

An application of Pollaczek polynomials to the spectral decomposition of a Toeplitz operator on H^2 is found in:

Rosenblum, M. – Self-adjoint Toeplitz operators and associated orthonormal functions, Proc. Amer. Math. Soc. **13**, 590-595 (1962).